A Post-Quantum Oblivious PRF from Isogenies

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Abstract. An oblivious pseudorandom function, or OPRF, is an important primitive that is used to build many advanced cryptographic protocols. Despite its relevance, very few post-quantum solutions exist.
In this work, we propose a novel OPRF protocol that is post-quantum, verifiable, round-optimal, and moderately compact.
Our protocol is based on a previous SIDH-based construction by Boneh, Kogan, and Woo, which was later shown to be insecure due to an attack on its one-more unpredictability.
We first propose an efficient countermeasure against this attack by redefining the PRF function to use irrational isogenies. This prevents a malicious user from independently evaluating the PRF. The SIDH-based construction by Boneh, Kogan, and Woo is also vulnerable to the recent attacks on SIDH. We thus demonstrate how to efficiently incorporate the countermeasures against such attacks to obtain a secure OPRF protocol. To achieve this, we also propose the first proof of isogeny knowledge that is compatible with masked torsion points, which may be of independent interest.
Lastly, we design a novel non-interactive proof of knowledge of parallel isogenies, which reduces the number of communication rounds of the OPRF to the theoretically-optimal two.
Putting everything together, we obtain the most compact post-quantum verifiable OPRF protocol.

Keywords: Oblivious Pseudorandom Functions · Isogenies · SIDH

1 Introduction

An oblivious pseudorandom function (OPRF) is a two-party protocol between a user and a server. The two parties obliviously evaluate a PRF on a user-controlled input with a secret key held by the server. After engaging in the protocol, the user learns only the output of the PRF on their chosen input, while the server does not learn anything, neither the user's input nor the output of the function. Oblivious PRFs can also satisfy a stronger property called *verifiability*: in a verifiable OPRF (vOPRF), the server initially commits to its secret key, and during the execution of the protocol it provides a proof that it has behaved honestly and it has used the previously committed secret key.

Oblivious PRFs have widespread applications in privacy-preserving constructions. For instance, the web browser Microsoft Edge uses an OPRF-based protocol to detect compromised passwords. Another practical use case of OPRFs is the privacy-preserving authorisation mechanism known as Privacy Pass [11]. OPRFs are also used within OPAQUE [22], a strong asymmetric password-authenticated key exchange. Overall, OPRFs are a fundamental tool for developing privacy-preserving solutions.

It is possible to build an OPRF using generic multi-party computation techniques, but such solutions can be inefficient, and they require more rounds of communication than what an ad-hoc construction can achieve. Indeed, highly-efficient and round-optimal (i.e., with two rounds) constructions exist based on Diffie-Hellman [20] or RSA blind signatures [9]. All such constructions are vulnerable to quantum attacks, and very few quantum-resistant OPRFs are reported in the literature. The first quantum-secure verifiable OPRF was proposed by Albrecht, Davidson, Deo and Smart [1]. The protocol is based on the learning-with-errors problem and the short-integer-solution problem in one dimension, and it only requires two rounds of communication. However, the construction can be characterized as a feasibility result, as a single OPRF execution requires communicating hundreds of gigabytes of data. The only other post-quantum solutions were proposed by Boneh, Kogan, and Woo [5]. The authors proposed two moderately-compact OPRFs based on isogenies, one relying on SIDH and one on CSIDH. The protocol based on CSIDH is a non-verifiable, three-round OPRF, which is obtained by combining a Naor-Reingold PRF [27] with a CSIDH-based oblivious transfer protocol [24] to make the PRF evaluation oblivious. The OPRF based on SIDH is verifiable, but requires an even higher number of communication rounds, since the verifiability proof is highly interactive. A later work by Basso, Kutas, Merz, Petit and Sanso [4] cryptanalyzed the SIDH-based OPRF by demonstrating two attacks against the one-more unpredictability of the protocol, i.e. it showed that a malicious user can recover sufficient information to independently evaluate the PRF on any input. The first attack is polynomial-time, but it can be easily prevented with a simple countermeasure; the second attack is subexponential but still practical, and the authors argue that there is no simple countermeasure against it. More recently, a series of works [7,25,28] developed an efficient attack on SIDH that also extends to the SIDH-based OPRF.

Contributions. In this work, we propose an OPRF protocol that is post-quantum secure, verifiable, round-optimal, and moderately compact (≈9 MB per execution), with a security proof in the UC framework [6] in the random-oracle model. To do so, we follow the same high-level approach as the SIDH-based OPRF by Boneh, Kogan, Woo [5], but with the following changes:

– We propose an efficient countermeasure against the one-more unpredictability attack by Basso, Kutas, Merz, Petit, and Sanso [4]. We modify the PRF definition, and in particular we use irrational isogenies to map the user's input to an elliptic curve. In this way, the information that allowed an attacker to independently evaluate the PRF is no longer defined over a field of

small extension. A malicious user may still attempt to carry out the attack from [4], but this would now require the attacker to work with points with exponentially many coordinates over the base field, which makes the attack infeasible. Besides preventing the attack, this change results in a smaller prime and a more compact protocol.
- We discuss how to integrate MSIDH, a recently-proposed countermeasure [15] against the SIDH attacks that relies on masked torsion, into the OPRF protocol. This requires using longer isogenies and a larger prime, but a series of optimizations allow us to maintain a reasonable communication cost. To integrate MSIDH, we also propose the first zero-knowledge proof of knowledge that can guarantee the correctness of an MSIDH public key, which may be of independent interest. The proof relies on splitting the masking value into three multiplicative shares: this enables the prover to build a commutative SIDH square and reveal a single edge, together with the torsion point images, without leaking any information about the witness. Repeating the process multiple times yields a proof with negligible soundness error.
- We propose a novel proof of knowledge that can guarantee that two isogenies are parallel, i.e. they are computed by applying the same secret key to two starting curves and torsion points. The protocol is obtained by evaluating two proofs of isogeny knowledge in parallel *with correlated randomness*. The proof can be proved zero-knowledge under a new yet mild assumption. Such a proof can be used by the server to show that it has used a previously committed secret key, which is the key ingredient to make the OPRF verifiable. Since the proof is a proof of knowledge, it can be made non-interactive through standard transformations; this makes the proposed OPRF the first isogeny-based OPRF to be round-optimal.

Paper organization. In Section 2, we introduce the main constructions needed for the rest of the paper. In Section 3, we present the OPRF ideal functionality, together with the security notions and assumptions needed to implement it. Section 4 presents novel countermeasures against the one-more unpredictability attack by Basso, Kutas, Merz, Petit and Sanso, while Section 5 discusses how to prevent the SIDH attacks, and Section 6 presents the new proof of parallel isogeny used to achieve verifiability. In Section 7, we put everything together to obtain the new OPRF protocol, estimate its communication complexity, and compare it with the existing solutions.

2 Preliminaries

2.1 SIDH

The Supersingular Isogeny Diffie-Hellman (SIDH) [19] is a key-exchange protocol based on isogenies between supersingular elliptic curves. For information on elliptic curves and isogenies, we refer the reader to [29]. The protocol parameters include a prime p of the form $p = ABf - 1$, where A and B

are smooth coprime integers and f a small cofactor, a supersingular curve E_0 defined over $\mathbb{F}_{p^2}$, and the basis P_A, Q_A and P_B, Q_B that span, respectively, $E_0[A]$ and $E_0[B]$. One party samples a secret key $a \xleftarrow{\$} \mathbb{Z}_A$, computes the isogeny $\phi_A : E_0 \to E_A \coloneqq E_0/\langle P_A + [a]Q_A \rangle$, and publishes $\mathsf{pk}_A = (E_A, R_A = \phi_A(P_B), S_A = \phi_A(Q_B))$. The second party, proceeds similarly by sampling a secret key $b \xleftarrow{\$} \mathbb{Z}_B$, computing $\phi_B : E_0 \to E_B \coloneqq E_0/\langle P_B + [b]Q_B \rangle$, and revealing $\mathsf{pk}_B = (E_B, R_B = \phi_B(P_A), S_B = \phi_B(Q_A))$. Then, both parties can agree on a shared secret by translating their secret isogeny to the starting curve in the other party's public key using the revealed torsion information. In other words, the first party computes $\phi'_A : E_B \to E_{AB} \coloneqq E_B/\langle R_B + [a]S_B \rangle$, and the second party computes $\phi'_B : E_A \to E_{BA} \coloneqq E_A/\langle R_A + [b]S_A \rangle$. The codomain curves E_{AB} and E_{BA} are isomorphic, and thus their j-invariant is the same and provides the shared secret of the key exchange. Note that it is possible to represent the points in the public keys more compactly than two x-coordinates, which requires $4 \log p$ bits. If the points are expressed in terms of a canonical basis , i.e. a basis deterministically computed from the curve, their coefficients only require $4 \log A$ or $4 \log B$ bits [10]. In the rest of the paper, we write $P, Q = \mathcal{B}_N(E)$ for a canonical basis of order N on E. We also refer to the setup described above as the *SIDH square* (E_0, E_A, E_B, E_{AB}) with edges $(\phi_A, \phi_B, \phi'_A, \phi'_B)$. The Weil pairing between points $P, Q \in E[N]$, for some curve E and integer N, is denoted by $e_N(P, Q)$, and it satisfies the property $e_N(\phi(P), \phi(Q)) = e_N(P, Q)^{\deg \phi}$.

The SIDH attacks. The security of the SIDH protocol hinges on the hardness of recomputing the secret isogenies given the public key. While the problem of finding an isogeny between two curves is believed to be hard, the presence of torsion point images in SIDH makes it easier since more information is revealed about the secret isogeny. In a series of works by Castryck and Decru [7], Maino, Martindale, Panny, Pope, and Wesolowski [25], and Robert [28], the authors propose a polynomial-time algorithm that can compute an isogeny of smooth degree d given the domain and codomain curves, the degree d, and the image of a torsion basis of order at least $\sqrt{d}$. This fully breaks the SIDH key exchange and all protocols based on it. Some countermeasures have been proposed [15], based on either secret-degree isogenies or on masked torsion images. We discuss these approaches in the context of the OPRF protocol in Section 5.

2.2 The OPRF construction by Boneh, Kogan, Woo

Boneh, Kogan, and Woo [5] introduced a verifiable OPRF protocol based on SIDH, which uses a prime p of the form $p = N_{\mathsf{M}} N_{\mathsf{B}} N_{\mathsf{K}} N_1 N_2 f - 1$, where the values N_i are coprime smooth integers and f is a small cofactor. Initially, the server commits to its key k by publishing the curve E_C obtained as the codomain of the N_{K}-isogeny starting from $\tilde{E}$ with kernel $\langle \tilde{P} + [k]\tilde{Q} \rangle$, where the values $\tilde{E}, \tilde{P}, \tilde{Q}$ are protocol parameters. The commitment also includes a zero-knowledge proof π_C of the correctness of the computation. Then, to evaluate the PRF on input $m \in \mathcal{M}$, where $\mathcal{M}$ defines the input space, the user computes an

isogeny ϕ_m of degree N_{M} by hashing the input with a function H and computing $\phi_m : E_0 \to E_m \coloneqq E_0/\langle P + [H(m)]Q\rangle$, where the curve E_0 and the points P, Q are also protocol parameters. Then, the user blinds the curve E_m by computing a second isogeny $\phi_b : E_m \to E_{mb}$ of degree N_{B}. The user sends the curve E_{mb} and the torsion images $R_K = \phi_b \circ \phi_m(P_K), S_K = \phi_b \circ \phi_m(Q_K)$ to the server, where the points P_K, Q_K are also protocol parameters of order N_{K}. The user also provides a non-interactive zero-knowledge proof that torsion information was honestly computed. The server validates the proof, computes the isogeny $\phi_k : E_{mb} \to E_{mbk} \coloneqq E_{mb}/\langle R_K + [k]S_K\rangle$ based on its secret key k, and sends the curve E_{mrk}, the image $\phi_k(E_{mb}[N_{\mathsf{B}}])$, and a non-interactive zero-knowledge proof of correctness to the user. Then, the server and the user engage in an interactive protocol where the server proves that the isogeny ϕ_k has used the same key k as the committed value. If the user is convinced, they use the provided torsion information to undo the blinding isogeny, i.e. to compute the translation of the dual of the blinding isogeny, to obtain the curve E_{mk}. The output of the OPRF is then the hash $H\big(m, j(E_{mk}), (E_C, \pi_C)\big)$. The main exchange, without the commitments and the proofs, is represented in Fig. 1.

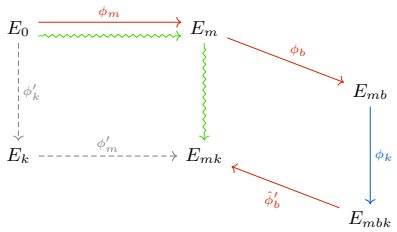

Fig. 1: The OPRF construction by Boneh, Kogan, and Woo. The protocol, without the relevant zero-knowledge proofs, is represented by the solid lines: the isogenies $\longrightarrow$ (ϕ_m, ϕ_b, $\hat{\phi}'_b$) are computed by the client, while the isogeny $\longrightarrow$ (ϕ_k) is computed by the server. The isogenies $\rightsquigarrow$ represent the PRF evaluation, and the isogenies $\dashrightarrow$ are relevant to the attack presented in [4]. The figure is based on Fig. 1 of [4].

2.3 The BKMPS attacks

Basso, Kutas, Merz, Petit, and Sanso [4] proposed two attacks against the one-more unpredictability of the OPRF protocol by Boneh, Kogan, Woo [5] OPRF. In the first attack, an attacker who acts as a malicious user engages in the OPRF with a message isogeny ϕ_m with kernel generator a point M, of order ℓ^e. The attacker repeats the process with message isogenies with kernel generators $[\ell]M, [\ell^2]M, \ldots, [\ell^e]M$. The outputs of the PRF are the curves that lie on the isogeny path of $\phi'_m : E_k \to E_{mk}$ (see Fig. 1), which allows the attacker to compute a generator of the kernel of such isogeny. The recomputed generator is

a scalar multiple $\phi'_k(M)$, where ϕ'_k is the isogeny parallel to the server's secret isogeny, i.e. its kernel is generated by $P_k + [k]Q_k$. By repeating this process three times with points M_1, M_2 and $M_3 \coloneqq M_1 + M_2$ (where M_1 and M_2 are linearly independent), the attacker obtains

$$R \coloneqq [\alpha]\phi'_k(M_0), \quad S \coloneqq [\beta]\phi'_k(M_1),$$
$$T \coloneqq [\gamma]\phi'_k(M_3) = [\gamma/\alpha]R + [\gamma/\beta]S,$$

for some unknown values α, β, γ. By expressing T in terms of R and S, the attacker obtains the values γ/α and γ/β and the points $R' \coloneqq [\gamma/\alpha]R = [\gamma]\phi'_k(M_0)$ and $S' \coloneqq [\gamma/\beta]S = [\gamma]\phi'_k(M_1)$. Finally, the attacker can evaluate the PRF on any input m by computing $E_K/\langle R' + [H(m)]S' \rangle$. The attack runs in polynomial time, but it crucially relies on using message isogenies ϕ_m of varying degree. The attack can be thwarted by server checking the order of the isogeny ϕ_m, which is possible because of the proof of knowledge provided by user.

The authors of [4] also propose a second attack that cannot be easily prevented. The attack proceeds in a similar way to the previous one, but the malicious user uses only isogenies of full degree. To obtain the curves on the path of ϕ_m, the attacker needs to find the middle curve between two PRF outputs. This introduces a trade-off between the complexity of the attack and the number of queries. Minimizing both yields a subexponential yet practical attack on the one-more unpredictability of the protocol.

3 Oblivious pseudorandom functions

The security properties of an OPRF can be hard to define. Oblivious pseudorandom functions were originally proposed by Naor and Reingold [27], who defined an OPRF via an ideal functionality. Subsequent work [16,23] defined OPRFs in terms of the two-party computation $(k, x) \mapsto (\perp, f(k, x))$, but such a definition has several drawbacks. On one side, it is hard to build protocols that satisfy such a definition, because the security proof would require extracting the user's input, while at the same the definition is not secure enough, because it does not guarantee any security under composability. Since OPRFs are mainly used as building blocks in larger protocols, such a security guarantee is highly needed. For these reasons, Jarecki et al. [20,22] proposed new definitions in the UC framework [6]. To avoid extracting the user's input, the ideal functionality introduces a ticketing system that increases a counter when the PRF is evaluated and decreases the counter when the user receives the PRF output. This captures the idea that a malicious user should learn only the PRF output for one input for each interaction. This results in the definition of Fig. 2, which is based on the definitions by Jarecki et al. [20,21,22].

3.1 Security assumptions

To prove that the OPRF protocol we propose implements the functionality of Fig. 2, we will make use of the properties listed in this section. Since our protocol

Parameters: The PRF output ℓ, polynomial in the security parameter λ.

Convention: For every identifier S, the counter $\mathsf{tx}[S]$ is initially set to zero. For every value $\pi \in \{0,1\}^*$ and $x \in \{0,1\}^*$, the value $F(\pi, x)$ is initially undefined, and whenever such a value is referenced, the functionality assigns a random ℓ-bit string $F(\pi, x) \xleftarrow{\$} \{0,1\}^\ell$.

Initialization

- On message INIT from party S, forward (INIT, S) to the adversary $\mathcal{A}$.
- On message (PARAM, S, π) from adversary $\mathcal{A}$, and if $\mathsf{param}[S]$ is undefined, then set $\mathsf{param}[S] = \pi$.

Evaluation

- On message (EVAL, S, x) from $P \in \{U, \mathcal{A}\}$, record $\langle P, x\rangle$ and forward the message (EVAL, P, S) to $\mathcal{A}$.
- On message SERVERCOMPLETE from server S, send $(\text{SERVERCOMPLETE}, S)$ to $\mathcal{A}$ and increment $\mathsf{tx}[S]$.
- On message $(\text{USERCOMPLETE}, P, \pi)$ from $\mathcal{A}$, retrieve the record $\langle P, x\rangle$, delete it from the list of records, and decrement $\mathsf{tx}[S]$ if there exists an honest server S such that $\mathsf{param}[S] = \pi$; abort if no such record exists or if $\mathsf{tx}[S] = 0$. Then, send $(\text{EVAL}, \pi, F(\pi, x))$ to P.

Fig. 2: The $\mathcal{F}_{\mathsf{vOPRF}}$ functionality.

and security proof follows the same high-level structure as that of the OPRF protocol by Boneh, Kogan, and Woo [5], these properties are also based on those of the augmentable commitment framework proposed in [5]. Unlike [5], we avoid the abstraction of augmentable commitments due to its restrictiveness (the counteremasures of Section 4 would not be possible within that framework), and we prefer an explicit description throughout this work.

Correctness. Firstly, we require the OPRF to be correct, i.e. the output of the protocol is the output of function that deterministically depends only on the user's input and the server's secret key. In other words, we want that the blinding process that guarantees the obliviousness of the user's input does not affect the final output. In the context of our protocol, we want that the unblinding isogeny undoes the effect of the blinding isogeny. This is contained in the following lemma, whose proof follows from the correctness of the SIDH protocol [19].

Lemma 1 (Correctness). *Let p be a prime of the form $p = N_B N_K f - 1$, where N_B, N_K, f are smooth coprime integers. Let E_0 be a supersingular elliptic curve defined over $\mathbb{F}_{p^2}$ and let P_B, Q_B and P_K, Q_K be respectively a basis of $E_0[N_B]$ and $E_0[N_K]$. Let also b and k be two values in $\mathbb{Z}_{N_B}$ and $\mathbb{Z}_{N_K}$. Then, consider the*

isogenies

$$\begin{aligned}\phi_B : E_0 \to E_B &\coloneqq E_0/\langle P_B + [b]Q_B\rangle,\\ \phi_K : E_0 \to E_K &\coloneqq E_0/\langle P_K + [k]Q_K\rangle,\\ \phi'_k : E_B \to E_{BK} &\coloneqq E_B/\langle \phi_B(P_K) + [k]\phi_B(Q_K)\rangle.\end{aligned}$$

If R_B, S_B is a basis of $E_B[N_B]$ and the values b_0, b_1 satisfy $\ker \hat{\phi}_B = \langle [b_0]R_B + [b_1]S_B\rangle$, then we have

$$j\left(E_{BK}/\langle [b_0]\phi'_k(R_B) + [b_1]\phi'_k(S_B)\rangle\right) = j(E_K).$$

Input hiding. To ensure that the OPRF is oblivious, we want that the server does not learn the user's input. That holds in the strongest sense, i.e. the server should not learn the user's input even when the input is randomly chosen between two inputs *chosen by the server*. In other words, the user must apply a blinding step that fully hides the chosen input. In the context of isogenies, we want the following problem to be hard.

Problem 1. Let p be a prime of the form $p = N_\mathsf{B} N_\mathsf{K} f - 1$, where $N_\mathsf{B} N_\mathsf{K}, f$ are smooth coprime integers. Let E_0 and E_1 be two supersingular elliptic curves defined over $\mathbb{F}_{p^2}$ and chosen by the adversary, and let P_0, Q_0 and P_1, Q_1 be a basis of $E_0[N_\mathsf{K}]$ and $E_1[N_\mathsf{K}]$, respectively, such that $e_{N_\mathsf{K}}(P_0, Q_0) = e_{N_\mathsf{K}}(P_1, Q_1)$. Let i be a random bit, i.e. $i \xleftarrow{\$} \{0, 1\}$, and B a random point in $E_i[N_\mathsf{B}]$, and write $\phi : E_i \to E' \coloneqq E_i/\langle B\rangle$. Output i given E' and $f_{\mathsf{aux}}(\phi(P_i), \phi(Q_i))$, where the latter is some auxiliary torsion information.

The hardness of the problem clearly depends on the function f_{aux}; if the torsion images were directly revealed, Problem 1 would be easy due to the SIDH attacks. We thus delay describing the function f_{aux} until Section 5, where we discuss the SIDH countermeasures and choose f_{aux} to reveal the values $\phi(P_i), \phi(Q_i)$, both scaled by the same unknown value. In the section, we also state the variant of the Decisional Isogeny problem that Problem 1 reduces to.

One-more unpredictability. A key property of an OPRF is that the user learns the output of the PRF only on its input of choice. That means that a malicious user should not learn the output on more inputs than the number of OPRF executions. The BKMPS attack [4] on the OPRF by Boneh, Kogan, and Woo [5] targets the one-more unpredictability, since it shows that a malicious user can extract enough information to independently evaluate the OPRF on any input of their choice. We propose an efficient countermeasure against the one-more unpredictability attack in the next section; we thus delay until then a formalization of the isogeny-related assumption (see Problem 4) we need to guarantee the one-more unpredictability of the OPRF protocol.

Commitment binding. At the beginning of the OPRF protocol, the server commits to a secret key k, so that during each OPRF execution it can prove that the same key was used. To guarantee verifiability, we want a commitment scheme with an associated proof of input reuse. We propose to commit to a key k by fixing a special curve $\tilde{E}$ with a basis $\tilde{P}, \tilde{Q}$ of $\tilde{E}[N_{\mathsf{K}}]$ and revealing $j(\tilde{E}/\langle \tilde{P} + [k]\tilde{Q}\rangle)$. The proof of input reuse, which in the context of isogenies becomes a proof of parallel isogenies, is presented in Section 5.2. To guarantee that the commitment is binding, we want that the following problem to be hard.

Problem 2 (Collision finding problem). Let E_0 be a supersingular elliptic curve of unknown endomorphism ring. Find two distinct isogenies $\phi_0 : E_0 \to E$ and $\phi_1 : E_0 \to E'$ such that $j(E_0) = j(E_1)$.

Problem 2 has been studied in the context of the CGL hash function [8], and it has been shown to be heuristically equivalent to the following problem, which underpins every isogeny-based protocol [13].

Problem 3 (Endomorphism Ring problem). Let E be a supersingular elliptic curve. Find its endomorphism ring $\mathrm{End}(E)$.

4 Countermeasures against the one-more unpredictability attack

The original protocol by Boneh, Kogan and Woo starts by mapping an input m to an isogeny ϕ_m. If we denote with N_{M} the torsion space dedicated to the message, the protocol fixes a basis P, Q of $E_0[N_{\mathsf{M}}]$ and computes the isogeny ϕ_m given by

$$\phi_m : E_0 \to E_0/\langle P + [H(m)]Q\rangle =: E_m, \tag{1}$$

where $H(\cdot)$ maps the message m onto an element of $\mathbb{Z}_{N_{\mathsf{M}}}$.

The subexponential attack [4] recovers the image P_k, Q_k of the torsion basis P, Q, up to scalar multiplication, under the secret isogeny $\phi'_k : E_0 \to E_k$. With such information, the attacker can evaluate the PRF on any input of their choice. The output curve of the PRF is the curve computed as $E_k/\langle P_k + [H(m)]_k\rangle$. The attack is subexponential, and it is possible to obtain λ bits of security if the isogeny ϕ_m has degree 2^{λ^2} (this can be reduced if we limit the number of queries the attacker can make). This would require using very long isogenies (the degree would be $2^{16,384}$ for $\lambda = 128$) and very large primes.

Instead, in this section we propose a novel and efficient countermeasure that sidesteps these issues. Our main idea is to accept that an attacker may recover the curve E_k and points P_k, Q_k on it, but to prevent those points from being sufficient to evaluate the desired isogeny. To do so, we require that the isogeny ϕ_m has an irrational kernel, i.e. its kernel is defined over a sufficiently-large extension field. Such an isogeny can be efficiently computed as a composition of rational isogenies. More formally, assume that $N_{\mathsf{M}} = \ell^e$, and e is the highest power of ℓ that divides $p + 1$. Then, given an input $m \in \mathcal{M}$, we compute the isogeny ϕ_m in the following way:

1. We first map the message m to two elements in $\mathbb{Z}_{\ell^e}$ through two hash functions H_0, H_1 that are collision resistant. We thus have $m_0 = H_0(m)$ and $m_1 = H_1(m)$.
2. Given the starting curve E_0 and two points P_0, Q_0 spanning $E_0[\ell^e]$, we compute the isogeny $\phi_0 : E_0 \to E_1 \coloneqq E_0/\langle P_0 + [m_0]Q_0 \rangle$.
3. We determine a canonical basis P_1, Q_1 of $E_1[\ell^e]$ and compute the isogeny $\phi_1 : E_1 \to E_m \coloneqq E_1/\langle P_1 + [m_1]Q_1 \rangle$,
4. The isogeny $\phi_m : E_0 \to E_m$ is the composition $\phi_1 \circ \phi_0$.

An attacker may still try to apply the one-more unpredictability attack. In the original case, the attacker recovers three isogenies from E_k to E_{mk} and they combine their kernel generators to obtain the image points P_k, Q_k. In the proposed construction, the attacker can still recover three (or more) isogenies from E_k to E_{mk}. However, the kernel generators of these isogenies are points of order ℓ^{2e}, and thus they are defined only over the extension field $\mathbb{F}_{p^{2\ell^e}}$. This is an exponentially large field, and even just representing such a point—let alone doing any computation—would be exponential in the security parameter. To guarantee security, it is important that the degree of ϕ_m is a prime power. If the degree were a product of prime powers, it is possible to represent a large extension by working over several smaller extensions because of the Chinese Remainder Theorem. This can reduce the complexity of working over a large extension and thus reduce the security of the proposed countermeasures.

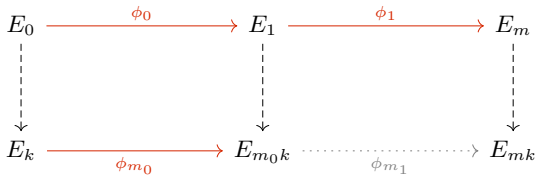

Fig. 3: Summary of the proposed countermeasure (this does not depict the blinding/unblinding phase). Isogenies in red are known or can be computed by the attacker, isogenies in black are unknown to the attacker, and the dotted isogeny represents the missing isogeny that the attacker needs to compute to succeed in the attack.

The attacker can work with the kernel generators of only the first half of the isogenies and obtain a basis P_k, Q_k of order ℓ^e (see Fig. 3). This allows them to evaluate the first isogeny ϕ_{m_0} to obtain the curve E_{m_0k} for any message m. However, the attacker has no way of computing the remaining isogeny ϕ_{m_1}. To do so, the attacker would need to map the canonical basis on E_1 to E_{m_0k}, which does not seem to be possible without knowing the server secret key. Alternatively, the attacker could map the points P, Q and P_k, Q_k under the isogenies ϕ_0 and ϕ'_k. At least one of the image points on each curve has full order, and the point of full order on E_{m_0k} is the image of the point of full order on E_1. This suggest such

an approach could be used to find a basis, but the second point on each curve is always a scalar multiple of the first point[1]. Hence, guessing the remaining point has exponential complexity ℓ^e. Lastly, the attacker cannot use a similar strategy as the one-more unpredictability attack to recover a basis on $E_{m_0 k}$ because the curve $E_{m_0 k}$ depends on the message m. It thus changes at every interaction, and it is hard for an attacker to find two messages that have the same first curve E_1 and $E_{m_0 k}$ since we assume that the hash function H_0 is collision-resistant. Note that we require H_0 and H_1 to be collision-resistant, but we conjecture that only H_0 needs to be. Overall, the knowledge of $E_{m_0 k}$ does not help the attacker learn any information on the curve E_{mk}, which successfully prevents the the one-more unpredictability attack.

Optimizations. We can extend this approach to obtain a more compact protocol. Rather than limiting ourselves to two isogenies, we can extend this to an arbitrary number I. We obtain the optimal case when I is maximal, i.e. when $\deg \phi_m = \ell^I$.

Let H_i be I distinct random oracles for every $i \in \{1, \ldots, I\}$. Then, given an input m and a starting curve E_0, the isogeny ϕ_m and the curve E_m by computing an isogeny ϕ_i determined by $H_i(m)$, generating a canonical basis on the codomain curve, and repeating the process I times (see Algorithm 1). We refer to this hashing as $\mathcal{H}_I(x)$, and in the rest of the paper, we write $(\phi_m, E_m) = \mathcal{H}_I(x)$ to refer to the function in Algorithm 1; we also write $[P_0, P_1, \ldots, P_{I-1}]_{E,N}$ to denote a list points of order N where the point P_0 belongs to E, and the point P_i belongs to $E_i \coloneqq E_{i_1}/\langle P_{i-1} \rangle$. We refer to this as a *sequence*, whose associated isogeny is the composition of the isogenies $E_i \to E_i/\langle P_i \rangle$.

Algorithm 1 Function $\mathcal{H}_I$ mapping the input m to the curve E_m

1: **for** $i \leftarrow 0$ to $I - 1$ **do**
2: Set $m_i = H_i(m)$ and $P_i, Q_i = \mathcal{B}_M(E)$;
3: Compute $\phi_i : E_i \to E_{i+1} \coloneqq E_i/\langle P_i + [m_i]Q_i \rangle$;
4: Set $\phi_m = \phi_{I-1} \circ \ldots \circ \phi_0$ and $E_m = E_{I-1}$;
5: **return** ϕ_m, E_m;

This technique to compute message isogenies results in a more compact OPRF protocol because only the shorter isogenies ϕ_i need to be defined over $\mathbb{F}_{p^2}$; thus, using more isogenies can result in a smaller prime p while maintaining the same degree of the isogeny ϕ_m. However, this approach has also a security advantage: an attacker can use the BKMPS attack to recover the image of basis on E_k, which could potentially be used to recover the isogeny between E_0 and E_k using the SIDH attacks. While this could be avoided by picking a sufficiently long isogeny ϕ_k, choosing $I = e$, i.e. setting parameters such that only isogenies

[1] If $\ker \phi = \langle P + \alpha Q \rangle$, it follows that $\phi(P) = -\alpha\phi(Q)$.

of degree ℓ have a kernel defined over $\mathbb{F}_{p^2}$, ensures that an attacker obtains only a basis of very small order, which prevents the attack altogether.

A new assumption. We proposed a modified protocol that prevents the existing one-more unpredictability attacks. As in the original construction, the one-more unpredictability of the resulting protocol relies on the hardness of a novel problem, which is the following.

Problem 4 (One-more unpredictability). Let p be a prime of the form $p = N_{\mathsf{M}} N_{\mathsf{K}} f - 1$, where N_{M} and N_{K} are smooth coprime integers, and f a cofactor. Let $\mathcal{H}_I$ be a function as in Algorithm 1. Let E_0 be a supersingular curve defined over $\mathbb{F}_{p^2}$, and let K be a point on E_0 of order N_{K}. Write ϕ_K for the isogeny $\phi_K : E_0 \to E_K \coloneqq E_0/\langle K \rangle$. Given the curves E_0, E_K and an oracle that responds to the following queries:

- $\mathsf{challenge}$: returns a random sequence $[M_0, \ldots, M_{I-1}]_{E_0, N_{\mathsf{M}}}$,
- $\mathsf{solve}([V_0, \ldots, V_{I-1}]_{E_0, N_{\mathsf{M}}})$: returns $j(E_V/\langle \phi_V(K) \rangle)$, where ϕ_V is the isogeny associated to the input sequence,
- $\mathsf{decide}(i, j)$: returns true if j is equal to the output of a solve query with input the response of the i-th $\mathsf{challenge}$ query, and false otherwise,

For any value n, produce n pairs (i, j) such that $\mathsf{decide}(i, j) = \mathsf{true}$ with less than n solve queries.

The problem is based on Game 12 of [5], but compared to it, this game involves multiple points during the $\mathsf{challenge}$ and solve query to abstract the behavior described in the previous section. Moreover, the problem includes the countermeasures against the polynomial time attack of [4], i.e. the attacker can only query points of the correct order. This can be replicated in the OPRF setting by checking the order of the isogenies in the proof of isogeny knowledge. We included these countermeasures to prevent possible attacks since they are inexpensive. However, we conjecture that the problem remains hard even if the adversary is allowed to submit solve queries with points of arbitrary order. Furthermore, the problem remains hard after the SIDH attacks since it does not involve exchanging any torsion points.

Countermeasure costs. We briefly discuss the impact of the proposed countermeasures on the performance of the OPRF protocol. Firstly, we need to determine the parameters ℓ, e, and I. Keeping in mind the possible SIDH attacks based on the recovered torsion on E_k, we choose I to be maximal. We require that the degree of the message isogeny is about $\approx 2^{5/2\lambda}$ to prevent the attack proposed in [26]. Hence, we set $e = 1$ and $I = \log_\ell(2^{5/2\lambda})$.

The message component N_{M} can then be chosen to be ℓ, to ensure that isogenies of degree ℓ have kernel in $\mathbb{F}_{p^2}$, or one, if torsion points of order ℓ are defined over a small extension field. In the latter case, the prime p does not need

to change to allow computations of the message isogeny. In both cases, not only do the proposed countermeasures protect against existing attacks, but also they reduce the prime size leading to a more compact and efficient protocol.

5 Countermeasures against the SIDH attacks

The recent series of attacks by Castryck and Decru [7], Maino, Martindale, Panny, Pope, and Wesolowski [25], and Robert [28] exploits torsion-point information to break SIDH. These attacks trivially translate to the OPRF, where any third party can recover both the user's hashed input (which breaks obliviousness) and the server's secret key. In this section, we discuss how to adapt the existing SIDH countermeasures to work in the OPRF setting. After modifying the main exchange, we propose a novel proof of isogeny knowledge that works together with the countermeasures, which may be of independent interest since it is the first proof to prove the correctness of torsion point images in the SIDH-with-countermeasure setting. This proof can be used together with the patched SIDH to obtain a post-quantum non-interactive key-exchange.

Combining the countermeasures together with the novel proof of torsion point correctness, we obtain an SIDH-based OPRF that is resistant against the SIDH attacks. While the countermeasures impose larger parameters, the resulting protocol remains the most compact post-quantum vOPRF.

5.1 Protecting the exchange

Thus, to guarantee the security of the SIDH-based OPRF we need to rely on the masked-torsion countermeasure, as in masked SIDH (MSIDH) [15]. Consider an isogeny $\phi : E \to E'$ of degree d, with a basis P, Q of $E[n]$, for some n coprime with d. Given the images $P' = \phi(P), Q' = \phi(Q)$, a second party can compute the pushforward of an isogeny with kernel $\langle P + [x]Q \rangle$ as the isogeny with kernel $\langle P' + [x]Q' \rangle$. Thus, it is possible to reveal $[\alpha]P', [\alpha]Q'$, for some random α coprime with the torsion order n, without affecting the correctness of the protocol. However, this leaks the value α^2 from the Weil pairing, since $e_n([\alpha]P', [\alpha]Q') = e_n(P, Q)^{\alpha^2 \deg \phi}$. To ensure that the attacker cannot recover the value α, MSIDH requires that any value has at least 2^λ square roots modulo n, hence n needs to be the product of at least λ prime powers. This, however, is not enough to guarantee security, as an attacker can guess the correct square root modulo some $n' \mid n$ with $n' > d^{1/2}$ in less than $\mathcal{O}(2^\lambda)$ guesses. It is thus also needed that $d > n'$, where n' is the product of the powers of the λ largest primes dividing n. From now on, we write $n = f_{\mathsf{MSIDH}}(\lambda, d)$ to denote the smallest value n that can guarantee λ bits of security when used in MSIDH with an isogeny of degree d. Lastly, the countermeasure analysis in [15] shows that MSIDH may be vulnerable when the starting curve has a small endomorphism. In our case, such an attack does not apply even if the OPRF starting curve E_0 has a known endomorphism ring with a small endomorphism ι. That is because the attack would need a small endomorphism on $E_m x$, but the composition of

the message and blinding isogeny $\phi_x \circ \phi_m$ is sufficiently long that the attack does not apply. Even considering the attack on the starting curve E_m (remember that in the security game the attacker can control the messages), does not help: if the attacker can guess the input message, the smallest endomorphism known on E_m is $\hat{\phi}_m \circ \iota \circ \phi_m$, which is too large for the attack to apply. Moreover, the server computes its isogeny starting from a curve E_{mx} that is sent by the user, which generally could be an avenue for attack since MSIDH is insecure for special starting curves. However, the user also submits a proof of knowledge of an isogeny of long degree between E_0 and E_{mx}: this guarantees that the smallest known endomorphism is again sufficiently large, which similarly prevents the attack against the server's isogeny.

We can now formulate the following problem, on whose hardness the input hiding property of the OPRF is based.

Problem 5 (Decisional MSIDH isogeny problem). Let E_0 be a supersingular elliptic curve, with a basis P, Q be of $E_0[n]$. Distinguish between the following distributions:

- (E_1, R, S), where E_1 is the codomain of a d-isogeny $\phi : E_0 \to E_1$, where d is coprime with n, and the points R, S are the masked images of P, Q, i.e. $R = [\alpha]\phi(P)$ and $S = [\alpha]\phi(Q)$ for some $\alpha \xleftarrow{\$} \mathbb{Z}_n^*$;
- (E_1, R, S), where E_1 is a random supersingular elliptic curve and the points R, S are a random basis of $E_1[n]$ such that $e(R, S) = e(P, Q)^{\alpha^2 d}$, for some value α.

The hardness of the problem clearly depends on the choices of n and d; the problem (conjecturally) requires $O(2^\lambda)$ operations to solve when $n > f_{\mathsf{MSIDH}}(\lambda, d)$, i.e. the product of the λ largest prime powers dividing n is smaller than $\sqrt{d}$.

Concrete cost. We have shown it is possible to protect the OPRF protocol from the SIDH attacks. Unfortunately, the proposed countermeasure do come at a significant cost. The degrees of the blinding isogeny and the server's isogeny are the same as in SIDH with the same countermeasures. At security level $\lambda = 128$, that corresponds to isogenies of degree $\approx 2^{2956}$. More generally, we see experimentally that the degree of the isogenies scales log-linearly in the security parameter with a constant of ≈ 6.7. We thus have that the degree of the blinding isogeny and the server's isogeny must be $\approx 2^{6.7\lambda \log \lambda}$ to guarantee the security of the protocol.

5.2 Adapting the proof of isogeny knowledge

In the previous section, we showed how it is possible to protect the OPRF against the SIDH attacks using masked torsion points. However, in the OPRF protocol both parties need to prove the correctness of their torsion images to prevent adaptive attacks and guarantee the verifiability of the execution. Thus, both

parties want to prove that their revealed torsion points were honestly generated, i.e. the two points are both scaled by the same value.

In this section, we propose a zero-knowledge proof of isogeny knowledge that can guarantee the correctness of torsion points up to a scalar, i.e. a proof for the following relation:

$$\mathcal{R}_{\mathsf{iso}} = \left\{ ((E_0, P_0, Q_0, E_1, P_1, Q_1), (\phi, \alpha)) \;\middle|\; \begin{array}{c} \phi : E_0 \to E_1 \text{ is a cyclic } d\text{-isogeny}, \\ P_1 = [\alpha]\phi(P_0), \\ Q_1 = [\alpha]\phi(Q_0). \end{array} \right\}.$$

In the literature, we can find two proofs of isogeny knowledge that also guarantee the correctness of torsion point images. The first proof constructs an SIDH square and explicitly maps the torsion images through all the sides of the square. This proof was proposed by Boneh, Kogan, and Woo [5] for the OPRF protocol, based on a previous idea by Galbraith [17]. The second proof [12], instead, is an extension of the simpler proof of isogeny knowledge by De Feo and Jao [19]. The first proof requires a larger prime, but the torsion images are explicitly mapped, which makes it well-suited to support masked torsion. We thus propose a new proof based on the same approach as [5] and [17], although with some notable differences. Building a more compact proof based on the second approach [12] remains an open problem.

The main idea is that the masking constant α can be split into three shares $\alpha = \alpha_1\alpha_2\alpha_3$. The prover can mask the torsion points with α_i when computing the i-th side of the SIDH square, so that the composition of the three isogenies, together with their masking values, forms a commutative diagram with the isogeny ϕ with masking value α. The proof remains zero-knowledge because each single value α_i is independent of α.

More formally, let E_0 and E_1 be supersingular elliptic curves with points $P_0, Q_0 \in E_0[n]$ and $P_1, Q_1 \in E_1[n]$. The prover wants to prove knowledge of a d-isogeny $\phi : E_0 \to E_1$ and a value $\alpha \in \mathbb{Z}_n$ such that $P_1 = [\alpha]\phi(P_0)$ and $Q_1 = [\alpha]\phi(Q_0)$. Let us assume $n = f_{\mathsf{MSIDH}}(\lambda, d)$, so that the isogeny ϕ is hard to extract from public information. The prover generates a random isogeny $\psi : E_0 \to E_2$ of degree s, where $s \approx n$ is a smooth number coprime with both n and d, and generates the SIDH square (E_0, E_1, E_2, E_3) with edges $(\phi, \psi, \phi', \psi')$. To guarantee soundness, the prover needs to show that ψ and ψ' are parallel: the prover thus generates a s-basis R_2, S_2 on E_2, maps it to E_3 to obtain R_3, S_3, and expresses the kernels of $\hat{\psi}$ and $\hat{\psi}'$ in terms of R_2, S_2 and R_3, S_3 with the same linear coefficients. The prover also splits α in three shares $\alpha = \alpha_1\alpha_2\alpha_3$ and maps the points P_0, Q_0 through ψ and ϕ' with masking values α_1 and α_2 to obtain the points

$$\begin{aligned} P_2 = [\alpha_1]\psi(P_0),\ Q_2 = [\alpha_1]\psi(Q_0), \\ P_3 = [\alpha_2]\phi'(P_2),\ Q_3 = [\alpha_2]\phi'(Q_2), \end{aligned}$$

which implies that P_3 and Q_3 also satisfy the relation

$$[\alpha_3]P_3 = \psi'(P_1),\ [\alpha_3]Q_3 = \psi'(Q_1).$$

Hence, the SIDH square commutes with respect to the points P_i, Q_i, i.e. if we restrict ourselves to the n-torsion, we have

$$[\alpha][s]\phi = [\alpha_3]\hat{\psi}' \circ [\alpha_2]\phi' \circ [\alpha_1]\psi.$$

Thus, the witness can be split into three components, and hence we obtain a proof with ternary challenges. The prover initially commits to the curves E_2, E_3 and the relevant points on them with a commitment scheme $\mathsf{C}(\cdot)$. Then, depending on the challenge, the prover responds with one edge of the SIDH square, the relevant curves and points, and the corresponding commitment openings. The proof is described in Fig. 4. Since each iteration has soundness error $2/3$, the proof must be repeated $-\lambda \log_{2/3}(2) \approx 1.71$ times to achieve a soundness error of $2^{-\lambda}$.

Remark 1. If the kernel of the isogeny ϕ is not defined over a small extension field, as in the case of the message isogeny, the proof can be computed by gluing together multiple SIDH squares, as shown in [3].

$\mathsf{P}_1((E_0, P_0, Q_0), (E_1, P_1, Q_1), \phi, \alpha)$:

Sample a random cyclic isogeny $\psi : E_0 \to E_2$ of degree s;
Construct the SIDH square $(E_0, E_1, E_2, E_3, \phi', \psi')$ on (ϕ, ψ);
Sample random units $\alpha_1, \alpha_2 \bmod n$ and set $a_3 \coloneqq \alpha/\alpha_1\alpha_2$;
Set $P_2, Q_2 \coloneqq [\alpha_1]\psi(P_1), [\alpha_1]\psi(Q_1)$, and $P_3, Q_3 \coloneqq [\alpha_2]\phi'(P_2), [\alpha_2]\phi'(Q_2)$;
Let R_2, S_2 be a basis of $E_2[d]$ and set $R_3, S_3 \coloneqq \phi'(R_2), \phi'(S_2)$;
Write $K = [a]R_2 + [b]S_2$ for K a random generator of $\ker \hat{\psi}$
Sample random strings $r_1, \ldots, r_7$;
return $\big(\mathsf{st}, \mathsf{C}(E_2, R_2, S_2, P_2, Q_2; r_1), \mathsf{C}(E_3, R_3, S_3, P_3, Q_3; r_2),$
$\mathsf{C}(a, b; r_3), \mathsf{C}(\phi'; r_4), \mathsf{C}(\alpha_1; r_5), \mathsf{C}(\alpha_2; r_6), \mathsf{C}(\alpha_3; r_7)\big)$.

$\mathsf{P}_2(\mathsf{st}, \mathsf{chall})$:

if $\mathsf{chall} == -1$ **then**
  return $((E_2, R_2, S_2, P_2, Q_2, r_1), (a, b, r_3), (\alpha_1, r_5))$;
else if $\mathsf{chall} == 0$ **then**
  return $((E_2, R_2, S_2, P_2, Q_2, r_1), (E_3, R_3, S_3, P_3, Q_3, r_2), (\phi', r_4), (\alpha_2, r_6))$;
else if $\mathsf{chall} == 1$ **then**
  return $((E_3, R_3, S_3, P_3, Q_3, r_2), (a, b, r_3), (\alpha_3, r_7))$;

$\mathsf{V}((E_0, P_0, Q_0), (E_1, P_1, Q_1), (\mathsf{com}_1, \ldots, \mathsf{com}_9), \mathsf{chall}, \mathsf{resp})$:

if $\mathsf{chall} == -1$ **then**
  $((E_2, R_2, S_2, P_2, Q_2, r_1), (a, b, r_3), (\alpha_1, r_5)) = \mathsf{resp}$;
  Check $\mathsf{com}_1 = \mathsf{C}(E_2, R_2, S_2, P_2, Q_2; r_1)$,
    $\mathsf{com}_3 = \mathsf{C}(a, b; r_3)$, $\mathsf{com}_5 = \mathsf{C}(\alpha_1; r_5)$;
  Let $\hat{\psi}$ be the isogeny with kernel $\langle [a]R_2 + [b]S_2 \rangle$;
  Check $\hat{\psi}$ is an s-isogeny from E_2 to E_0;
  Check $[\alpha_1 s]P_0 = \hat{\psi}(P_2)$ and $[\alpha_1 s]Q_0 = \hat{\psi}(Q_2)$;
else if $\mathsf{chall} == 0$ **then**
  $((E_2, R_2, S_2, P_2, Q_2, r_1), (E_3, R_3, S_3, P_3, Q_3, r_2), (\phi', r_4), (\alpha_2, r_6)) = \mathsf{resp}$;

Check $\mathsf{com}_1 = \mathsf{C}(E_2, R_2, S_2, P_2, Q_2; r_1)$,
$\mathsf{com}_2 = \mathsf{C}(E_3, R_3, S_3, P_3, Q_3; r_2)$,
$\mathsf{com}_4 = \mathsf{C}(\phi'; r_4)$, $\mathsf{com}_6 = \mathsf{C}(\alpha_2; r_6)$;
Check ϕ' is a d-isogeny from E_1 to E_2;
Check $R_3, S_3 = \phi'(R_2), \phi'(R_3)$;
Check $P_3, Q_3 = [\alpha_2]\phi'(P_2), [\alpha_2]\phi'(Q_2)$;
else if $\mathsf{chall} == 1$ **then**
$((E_3, R_3, S_3, P_3, Q_3, r_2), (a, b, r_3), (\alpha_3, r_7)) = \mathsf{resp}$;
Check $\mathsf{com}_2 = \mathsf{C}(E_3, R_3, S_3, P_3, Q_3; r_2)$,
$\mathsf{com}_3 = \mathsf{C}(a, b; r_3)$, $\mathsf{com}_7 = \mathsf{C}(\alpha_3; r_7)$;
Check $\langle R_3, S_3 \rangle = E_3[s]$;
Let $\hat{\psi}'$ be the isogeny with kernel $\langle [a]R_3 + [b]S_3 \rangle$;
Check $\hat{\psi}$ is an s-isogeny from E_3 to E_1;
Check $[\alpha_3 s]P_1 = \hat{\psi}'(P_3)$ and $[\alpha_3 s]Q_1 = \hat{\psi}'(Q_3)$;

Fig. 4: Interactive proof of knowledge for the relation $\mathcal{R}_{\mathsf{iso}}$.

We now sketch the proofs of correctness, three-special soundness and zero-knowledge. Given the similarity of the zero-knowledge proof with those in [5], the proofs also follow a similar approach.

Correctness. A honest prover always generates proofs that are accepted by the verifier. The verifier recomputes the same operations as the prover and checks that the outputs match. The only difference is in the $\mathsf{chall} = \pm 1$ cases, where the verifier computes the dual of ψ and ψ', which then introduces a factor s in the point equality check.

Three-special soundness. The protocol is three-special sound because there exists an extractor that extracts the witness given three accepting transcripts with the same commitments and different challenges. The isogeny ϕ can be computed by mapping the kernel of ϕ' (from $\mathsf{chall} = 0$) under the isogeny $\hat{\psi}$ (from $\mathsf{chall} = -1$). Since the isogenies ψ and ψ' are parallel (from all the challenges combined), this guarantees that ϕ is a d-isogeny from E_0 to E_1. The masking value α can be recomputed as the product of α_1, α_2, and α_3.

Zero-knowledge. We sketch a simulator that given a statement $(E_0, P_0, Q_0, E_1, P_1, Q_1)$ and a challenge chall can simulate a valid transcript without knowledge of the witness. For the case $\mathsf{chall} = -1$, the simulator behaves like an honest prover. For $\mathsf{chall} = +1$, the situation is similar: the simulator can compute a d-isogeny ψ', pick a random basis R_3, S_3 of $E_3[d]$ and a random value $\alpha_3 \in \mathbb{Z}_n^*$, and compute the values a, b and points P_3, Q_3 that pass verification. Note that the points R_3, S_3 are uniformly random among the bases of $E_3[d]$, and the value α_3 is uniformly random and independent of α; the simulated values are thus distributed as the honestly-generated ones. The case of $\mathsf{chall} = 0$ is more complicated: the simulator can sample a random curve E_2, generate a random basis

P_2, Q_2 of $E_2[n]$ that satisfies $e(P_2, Q_2) = e(P_0, Q_0)^{x^2 s}$ for some random x, pick a random d-isogeny $\phi' : E_2 \to E_3$, and compute the image points on E_3. In this case, the indistinguishability of the simulator's output is only computational. It is thus based on the conjectured hardness of the following problem, which is a modified version of the Decisional Supersingular Product (DSSP) problem introduced in [19].

Problem 6 (DSSP with Torsion (DSSPwT) problem). Given an isogeny $\phi : E_0 \to E_1$ and points $P_0, Q_0 \in E_0[n]$, where $n = f_{\mathsf{MSIDH}}(\lambda, d)$, distinguish between the following distributions:

- $\mathcal{D}_0 = \{(E_2, P_2, Q_2, E_3, \phi')\}$, where E_2 is the codomain of an s-isogeny $\psi : E_0 \to E_2$, the points P_2, Q_2 satisfy $P_2 = [\alpha]\psi(P_0)$, $Q_2 = [\alpha]\psi(Q_0)$ for some $\alpha \in \mathbb{Z}_n^*$, and $\phi' : E_2 \to E_3$ is a d-isogeny with kernel $\ker \phi' = \psi(\ker \phi)$.
- $\mathcal{D}_1 = \{(E_2, P_2, Q_2, E_3, \phi')\}$, where E_2 is a random supersingular curve with the same cardinality as E_0, P_2 and Q_2 are two random points of order n such that $e(P_2, Q_2) = e(P_0, Q_0)^s$, and the isogeny ϕ' is a d-isogeny between E_2 and E_3.

Note that [5] argues that a similar proof can only reveal one torsion point (either P_i or Q_i) at a time to prevent a distinguishing attack on the simulator. The attack they present relies on computing the Weil pairing between two points of coprime order, and thus their pairing is always one. The attack thus does not apply, and the simulated transcript remains indistinguishable under Weil pairing checks because the sampled points P_2, Q_2 are guaranteed to have the same pairing as the honestly-generated points. By revealing both points P_i and Q_i we obtain a significantly more efficient proof, since it has $1/3$ soundness rather than $1/6$.

Optimizations. For simplicity, the proof in Fig. 4 contains a schematic description of the protocol, but the proof can be made more efficient through a series of optimizations. In the commitment phase, the value α_2 is only revealed together with the isogeny ϕ', and thus they can be committed together. Note that we have the prover commit to ϕ' to make the proof online-extractable without recursion, which is necessary to achieve a proof in the UC model. For applications of this proof outside of the OPRF context, the prover can avoid committing to ϕ'. The masking values α_1 and α_3 are independent of α, even when considered together, because α_2 is uniformly random. They can then be committed together and revealed both in the response to challenges $\mathsf{chall} = \pm 1$. Since the commitment for a, b is also revealed when $\mathsf{chall} = \pm 1$, the values a, b, α_1, α_3 can all be committed together. When $\mathsf{chall} = -1$, the curve E_3 and the points P_3, Q_3 are not revealed, and thus learning α_3 does not provide any information. The same applies to α_1 when $\mathsf{chall} = +1$. This allow us to reduce the number of commitments to four.
To further reduce the communication between prover and verifier, the basis R_2, S_2 on E_2 can be chosen canonically, so that it can be recomputed from E_2. Moreover, for the challenge $\mathsf{chall} = -1$, the prover can avoid revealing the

curve E_2, the points P_2, Q_2 and the coefficients a, b by revealing instead a kernel generator of ψ. The prover can recompute E_2, P_2, Q_2 and obtain a, b by writing a kernel generator of $\hat{\psi}$ in terms of the canonical basis R_2, S_2. Normally, the recomputed a, b would not be the same as those computed by the verifier since they are not unique. The problem can be avoided by fixing a canonical way to compute the coefficients, such as prescribing that one of the two coefficients must be one, and that a must be one if both coefficients are invertible mod s. The same approach holds for $\mathsf{chall} = +1$, except that the points R_3, S_3 have to be revealed by the prover. In the case of the horizontal isogeny, the prover can avoid revealing E_3 and the points R_3, S_3 and P_3, Q_3. They can all be recomputed from the remaining values.

Concrete cost. Each repetition of the proof requires two commitments, which are 2λ-bit long and use a λ-bit long opener. When $\mathsf{chall} = -1$, the prover reveals one s-isogeny, a masking value, and two commitment openers, which requires $\log n + \log s + 2\lambda$ bits. When $\mathsf{chall} = +1$, the prover also reveals two torsion points of order s: if they are compressed as in [2], the response requires $5 \log s + \log n + 2\lambda$ bits. Lastly, for $\mathsf{chall} = 0$, the prover reveals a curve, a d-isogeny, two points of order n, a masking value, and three openers; thus, the answer requires $2 \log p + \log d + 5 \log n + 3\lambda$ bits.
Hence, if we assume $d \approx n \approx s \approx \sqrt[3]{n}$, an average proof where the three challenges appear equally requires $\approx 1.71\lambda(20/9 \log p + 7/3\lambda)$ bits, while a worst-case proof, with only $\mathsf{chall} = 0$ challenges, requires $\approx 1.71\lambda(4 \log p + 3\lambda)$ bits.

6 Verifiability

Oblivious PRFs can satisfy a stronger security property called *verifiability*. Informally, this guarantees that the server behaves honestly and always uses the same long-term static key. This is needed to guarantee the privacy of the user in those instances where the user may later reveal the output of the OPRF. A malicious server may behave "honestly" while also using different secret keys on different interactions. After learning the OPRF output of the user, the server can then test which secret key was used to produce that specific output and thus link the user to a specific user-server interaction.
The OPRF protocol by Boneh, Kogan, and Woo achieves verifiability by introducing three components. First, the server initially commits to a secret key k. The commitment is in the form of an elliptic curve $E_C \coloneqq E/\langle P + [k]Q \rangle$, where the curve E and the points P, Q are fixed parameters. Second, during the OPRF execution, the server provides a zero-knowledge proof that its computations used the same key as the one in the commitment. We refer to this proof as a *proof of parallel isogeny* (PoPI). Lastly, the server also provides two *proofs of isogeny knowledge* (PoIKs) that guarantee the correctness of the computations during both the commitment stage and the OPRF execution. The proof of parallel isogeny proposed by Boneh, Kogan, and Woo relies on the user and the server engaging in an SIDH exchange, where one of the sides is either the commitment

isogeny or the the secret server isogeny in the OPRF protocol. However, this proof is inherently interactive, and it requires five rounds of interaction. Moreover, the proof relies on multiple SIDH exchanges, and it is thus broken by the attacks on SIDH [7,25,28].

We avoid these issues by introducing a novel public-coin proof protocol of parallel isogeny. Since the proof does not rely on private randomness, we obtain a proof *of knowledge* that can be made non-interactive via the Fiat-Shamir transform [14] or the Unruh transform [30]. In the OPRF setting, we will rely on the latter to achieve the online-extractability without rewinding needed to get a proof in the UC model. Our main approach relies on executing two proofs of isogeny knowledge in parallel *with correlated randomness*. Since part of the randomness used is shared, we can obtain a proof of parallelness without needing additional computations.

Firstly, we formalize the notion of parallelness. We say that two d-isogenies $\phi : E_0 \to E_1$ and $\tilde{\phi} : \tilde{E}_0 \to \tilde{E}_1$ are parallel with respect to the bases $T_0, V_0 \in E_0[d]$ and $\tilde{T}_0, \tilde{V}_0 \in E'_0[d]$ if there exists coefficients $a, b \in \mathbb{Z}_d$ such that $\ker \phi = \langle [a]T_0 + [b]V_0 \rangle$ and $\ker \phi = \langle [a]\tilde{T}_0 + [b]\tilde{V}_0 \rangle$. This suggests that the parallelness relation that we are proving is the following:

$$\mathcal{R}_{\mathsf{par}} = \left\{ ((E_0, T_0, V_0, E_1, \tilde{E}_0, \tilde{T}_0, \tilde{V}_0, \tilde{E}_1), (k_0, k_1)) \;\middle|\; \begin{matrix} E_0/\langle [k_0]T_0 + [k_1]V_0 \rangle \cong E_1, \\ \tilde{E}_0/\langle [k_0]\tilde{T}_0 + [k_1]\tilde{V}_0 \rangle \cong \tilde{E}_1 \end{matrix} \right\}$$

However, as discussed before, we are combining several proofs together to obtain a larger proof that simultaneously proves knowledge of two isogenies and guarantees the two isogenies are parallel. We thus obtain a proof for the following relation, where we consider the case of a secret key with two coefficients for completeness. For practical reasons, the OPRF will fix $k_0 = 1$ without any loss of security.

$$\mathcal{R}^*_{\mathsf{par}} = \left\{ \begin{pmatrix} (E_0, T_0, V_0, P_0, Q_0, E_1, P_1, Q_1, \\ \tilde{E}_0, \tilde{T}_0, \tilde{V}_0, \tilde{P}_0, \tilde{Q}_0, \tilde{E}_1, \tilde{P}_1, \tilde{Q}_1), \\ (k_0, k_1, \alpha, \alpha') \end{pmatrix} \;\middle|\; \begin{matrix} \ker \phi = \langle [k_0]T_0 + [k_1]V_0 \rangle, \\ \ker \phi' = \langle [k_0]\tilde{T}_0 + [k_1]\tilde{V}_0 \rangle, \\ (E_0, P_0, Q_0, E_1, P_1, Q_1), (\phi, \alpha) \in \mathcal{R}_{\mathsf{iso}}, \\ (\tilde{E}_0, \tilde{P}_0, \tilde{Q}_0, \tilde{E}_1, \tilde{P}_1, \tilde{Q}_1), (\phi', \alpha') \in \mathcal{R}_{\mathsf{iso}} \end{matrix} \right\}$$

Now, let the curve $\tilde{E}_0$ with a d-basis $\tilde{T}_0, \tilde{V}_0$ be fixed protocol parameters. Using the same notation as before, assume that server has committed to its key (k_0, k_1) by publishing the codomain of the d-isogeny $\tilde{\phi}$ that has kernel $\langle [k_0]\tilde{T}_0 + [k_1]\tilde{V}_0 \rangle$. The server may also reveal some torsion information in its commitment, but as we will discuss later, this is not strictly needed. During the OPRF execution, the server receives a curve E_0 with a d-basis T_0, V_0 on it, and it computes $\phi : E_0 \to E_1 \coloneqq E_0/\langle [k_0]T_0 + [k_1]V_0 \rangle$. The server then wants to prove that it knows the isogenies ϕ and $\tilde{\phi}$ and that they are parallel.

If the server simply ran two instances of the PoIK from Fig. 4 in parallel, there would be no way to convince the prover that the isogenies are indeed parallel. If the proofs share the same challenges, i.e. the verifier sends the same challenges to both proofs, the server would respond with both ϕ' and $\tilde{\phi}'$ when $\mathsf{chall} = 0$. However, the isogenies ϕ' and $\tilde{\phi}'$ are parallel with respect to the bases $\psi(T_0), \psi(V_0)$

and $\tilde{\psi}(\tilde{T}_0), \tilde{\psi}(\tilde{V}_0)$ (where ψ is the vertical isogeny used in the proof of knowledge), which are not revealed in the proof. If we were to reveal them, the proof would not be zero-knowledge, because when $\mathsf{chall} = 0$, the verifier could recompute the secret isogeny ψ and $\tilde{\psi}$ through the SIDH attacks. Instead, we want to modify the proof to reveal different bases $T_2, V_2 \in E_2[d]$ and $\tilde{T}_2, \tilde{V}_2 \in \tilde{E}_2[d]$ such that ϕ' and $\tilde{\phi}'$ are parallel with regards to them, but also such that they do not reveal much information about ψ and $\tilde{\psi}$. We thus propose that the prover generates four random coefficients $w, x, y, z \in \mathbb{Z}_d$ such that $wz - xy \neq 0 \bmod d$, and computes T_2 and V_2 as the solution of

$$\psi(T_0) = [w]T_2 + [x]S_2, \quad \psi(V_0) = [y]T_2 + [z]V_2,$$

and similarly for $\tilde{T}_2$ and $\tilde{V}_2$. This is then secure, because the basis T_2, V_2 is uniformly random. Thus, for a single proof, this change only does not affect the security of the proof since no additional information is revealed. The rest of the proof needs to be modified to ensure that the process is followed correctly, i.e. we want the prover to reveal the values w, x, y, z together with ψ so that the verifier can verify the correctness of T_2 and V_2. The modified proof is denoted by $\mathcal{P}^*_{\mathsf{iso}}$, and it is represented explicitly in Fig. 5.

$\mathsf{P}_1^*\Big(((E_0, T_0, V_0, P_0, Q_0), (E_1, P_1, Q_1), \phi, \alpha), (w, x, y, z)\Big)$:
1-4: Same as P_1 in Fig. 4.
5: Set T_2, V_2 to satisfy $\psi(T_0) = [w]T_2 + [x]S_2, \psi(V_0) = [y]T_2 + [z]V_2$;
6-8: Same as P_1 in Fig. 4.

$\mathsf{P}_2^*(\mathsf{st}, \mathsf{chall})$:
1: **if** $\mathsf{chall} == -1$ **then**
2: **return** $((E_2, T_2, V_2, P_2, Q_2, r_1), (a, b, r_3), (\alpha_1, r_5), (w, x, y, z))$;
3: **else if** $\mathsf{chall} == 0$ **then**
4: **return** $((E_2, T_2, V_2, R_2, S_2, P_2, Q_2, r_1), (E_3, R_3, S_3, P_3, Q_3, r_2), (\phi', r_4), (\alpha_2, r_6))$;
5: **else if** $\mathsf{chall} == 1$ **then**
6: Same as P_2 in Fig. 4.

$\mathsf{V}^*\Big(\big(((E_0, T_0, V_0, P_0, Q_0), (E_1, P_1, Q_1), \phi, \alpha), \mathsf{com}, \mathsf{chall}, \mathsf{resp}\Big)$:
1: **if** $\mathsf{chall} == -1$ **then**
2: $((E_2, R_2, S_2, P_2, Q_2, r_1), (a, b, r_3), (\alpha_1, r_5), (w, x, y, z)) = \mathsf{resp}$;
3-6: Same as V in Fig. 4.
7: Check $\psi(T_0) = [w]T_2 + [x]S_2, \psi(V_0) = [y]T_2 + [z]V_2$;
8: **else**
9: Same as V in Fig. 4.

Fig. 5: Modified proof of knowledge for the relation $\mathcal{R}_{\mathsf{iso}}$ where the basis randomness is explicit. The expressions in magenta denote the changes from Fig. 4.

Now, if the prover executes the modified proof of isogeny knowledge for ϕ and $\tilde{\phi}$ in parallel, with the same challenges, and with the same values x, w, y, z, the isogenies $\phi', \tilde{\phi}'$ revealed when $\mathsf{chall} = 0$ are parallel when the isogenies $\phi, \tilde{\phi}$ are also parallel, as shown in the following lemma.

Lemma 2. *Let notation be as above. The isogenies $\phi, \tilde{\phi}$ are parallel if and only if the isogenies $\phi', \tilde{\phi}'$ are also parallel.*

Proof. Assume the isogeny ϕ has kernel $\langle [k_0]T_0 + [k_1]V_0 \rangle$ and the isogeny $\tilde{\phi}$ has kernel $\langle [\tilde{k}_0]\tilde{T}_0 + [\tilde{k}_1]\tilde{V}_0 \rangle$. The kernel of ϕ' is the image of the kernel of ϕ under ψ, i.e. $\ker \phi' = \psi(\ker \phi)$. Since $\ker \phi = \langle [k_0]T_0 + [k_1]V_0 \rangle$, it follows that

$$\ker \phi' = \langle [k_0]\psi(T_0) + [k_1]\psi(V_0) \rangle = \langle [wk_0 + yk_1]T_2 + [xk_0 + zk_1]V_2 \rangle.$$

Similarly, we obtain

$$\ker \tilde{\phi}' = \langle [w\tilde{k}_0 + y\tilde{k}_1]\tilde{T}_2 + [x\tilde{k}_0 + z\tilde{k}_1]\tilde{V}_2 \rangle.$$

Since the coefficients w, x, y, z were chosen such that the matrix $\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)$ is invertible, we obtain that

$$(k_0 = \tilde{k}_0) \wedge (k_1 = \tilde{k}_1) \iff (wk_0 + yk_1 = w\tilde{k}_0 + y\tilde{k}_1) \wedge (xk_0 + zk_1 = x\tilde{k}_0 + z\tilde{k}_1).$$

□

We can now use the proof $\mathcal{P}^*_{\mathsf{iso}}$ from Fig. 5 to construct our proof of parallel isogeny knowledge. The prover runs two such proofs in parallel, with the same randomness (w, x, y, z), and responds to the verifier's challenges with the responses of the individual proofs. The resulting proof is represented explicitly in Fig. 6. The security proofs follow closely those of the PoIK $\mathcal{P}_{\mathsf{iso}}$ in Section 5.2: correctness of $\mathcal{P}_{\mathsf{iso}}$ implies correctness of $\mathcal{P}_{\mathsf{par}}$, while the soundness of $\mathcal{P}_{\mathsf{par}}$ follows from the soundness of $\mathcal{P}_{\mathsf{iso}}$ and Lemma 2. The argument for zero-knowledge is also similar, but it is based on the hardness of the following problem, which takes into consideration that the two parallel instance partially share the same randomness.

Problem 7 (Double DSSP with Torsion (DDSSPwT) problem). Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be as in Problem 6. Given:

1. two d-isogenies $\phi : E_0 \to E_1$, $\tilde{\phi} : \tilde{E}_0 \to \tilde{E}_1$,
2. the points $T_0, V_0 \in E_0[d]$ and $\tilde{T}_0, \tilde{V}_0 \in \tilde{E}_0[d]$,
3. the points $P_0, Q_0 \in E_0[n]$ and $\tilde{P}_0, \tilde{Q}_0 \in \tilde{E}_0[n]$, where $n = f_{\mathsf{MSIDH}}(\lambda, d)$,

distinguish between the following distributions:

- $\mathcal{D}^*_0 = \left\{ \begin{matrix} (E_2, T_2, V_2, P_2, Q_2, E_3, \phi'), \\ (\tilde{E}_2, \tilde{T}_2, \tilde{V}_2, \tilde{P}_2, \tilde{Q}_2, \tilde{E}_3, \tilde{\phi}') \end{matrix} \right\}$, where the curves and the n-torsion points follow the $\mathcal{D}_0$-distribution, i.e. we have $(E_2, P_2, Q_2, E_3, \phi') \leftarrow \mathcal{D}_0$, and $(\tilde{E}_2, \tilde{P}_2, \tilde{Q}_2, \tilde{E}_3, \tilde{\phi}') \leftarrow \mathcal{D}_0$, and moreover

$$\begin{bmatrix} T_2 \\ V_2 \end{bmatrix} = B \begin{bmatrix} \psi(T_0) \\ \psi(V_0) \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \tilde{T}_2 \\ \tilde{V}_2 \end{bmatrix} = B \begin{bmatrix} \tilde{\psi}(\tilde{T}_0) \\ \tilde{\psi}(\tilde{V}_0) \end{bmatrix},$$

$\mathsf{P}_1((E_0, T_0, V_0, P_0, Q_0, E_1, P_1, Q_1), (\tilde{E}_0, \tilde{T}_0, \tilde{V}_0, \tilde{P}_0, \tilde{Q}_0, \tilde{E}_1, \tilde{P}_1, \tilde{Q}_1), k_0, k_1, \alpha, \alpha')$:
1: Sample random coefficients w, x, y, z such that $wz - xy \neq 0 \bmod d$;
2: Compute $\phi : E_0 \to E_0/\langle [k_0]T_0 + [k_1]V_0 \rangle \cong E_1$
3: Compute $\phi' : \tilde{E}_0 \to \tilde{E}_0/\langle [k_0]\tilde{T}_0 + [k_1]\tilde{V}_0 \rangle \cong \tilde{E}_1$
4: Run $\mathsf{P}_1^*((E_0, P_0, Q_0, E_1, P_1, Q_1), \phi, \alpha, (w, x, y, z))$ to get $\mathsf{st}, \mathsf{com}$;
5: Run $\mathsf{P}_1^*((\tilde{E}_0, \tilde{P}_0, \tilde{Q}_0, \tilde{E}_1, \tilde{P}_1, \tilde{Q}_1), \tilde{\phi}', \tilde{\alpha}', (w, x, y, z))$ to get $\tilde{\mathsf{st}}, \tilde{\mathsf{com}}$;
6: **return** $\big((\mathsf{st}, \tilde{\mathsf{st}}), (\mathsf{com}, \tilde{\mathsf{com}})\big)$;

$\mathsf{P}_2((\mathsf{st}, \tilde{\mathsf{st}}), \mathsf{chall})$:
1: **return** $(\mathsf{P}_2^*(\mathsf{st}, \mathsf{chall}), \mathsf{P}_2^*(\tilde{\mathsf{st}}, \mathsf{chall}))$;

$\mathsf{V}((E_0, T_0, V_0, P_0, Q_0, E_1, P_1, Q_1), (\tilde{E}_0, \tilde{T}_0, \tilde{V}_0, \tilde{P}_0, \tilde{Q}_0, \tilde{E}_1, \tilde{P}_1, \tilde{Q}_1),$
$(\mathsf{com}, \tilde{\mathsf{com}}), \mathsf{chall}, (\mathsf{resp}, \tilde{\mathsf{resp}}))$:
1: Set $v := \mathsf{V}^*((E_0, R, S, P_0, Q_0, E_1, P_1, Q_1), \mathsf{com}, \mathsf{chall}, \mathsf{resp})$;
2: Set $\tilde{v} := \mathsf{V}^*((\tilde{E}_0, \tilde{T}, \tilde{V}, \tilde{P}_0, \tilde{Q}_0, \tilde{E}_1, \tilde{P}_1, \tilde{Q}_1), \tilde{\mathsf{com}}, \mathsf{chall}, \tilde{\mathsf{resp}})$;
3: **if** $\mathsf{chall} == 0$ and $\phi', \tilde{\phi}'$ are *not* parallel wrt T_2, V_2 and $\tilde{T}_2, \tilde{V}2$ **then**
4: **return** $\perp$;
5: **return** $v \wedge \tilde{v}$;

Fig. 6: Interactive proof of knowledge for the relation $\mathcal{R}^*_{\mathsf{par}}$.

for some $B \in \mathrm{GL}_2(\mathbb{Z}_n)$, and ψ and $\tilde{\psi}$ being respectively the s-isogenies between E_0 and E_2 and $\tilde{E}_0$ and $\tilde{E}_2$ that are guaranteed to exist because of the $\mathcal{D}_0$ distribution;

– $\mathcal{D}_1^* = \left\{ \begin{matrix} (E_2, T_2, V_2, P_2, Q_2, E_3, \phi'), \\ (\tilde{E}_2, \tilde{T}_2, \tilde{V}_2, \tilde{P}_2, \tilde{Q}_2, \tilde{E}_3, \tilde{\phi}') \end{matrix} \right\}$, where the curves and the n-torsion points follow the $\mathcal{D}_1$-distribution, i.e. we have $(E_2, P_2, Q_2, E_3, \phi') \leftarrow \mathcal{D}_1$, and $(\tilde{E}_2, \tilde{P}_2, \tilde{Q}_2, \tilde{E}_3, \tilde{\phi}') \leftarrow \mathcal{D}_1$, and moreover the points T_2, V_2 and $\tilde{T}_2, \tilde{V}_2$ form a random basis of $E_2[d]$ and $\tilde{E}_2[d]$, respectively.

The proof $\mathcal{P}_{\mathsf{par}}$ is a proof of knowledge, and it can be made non-interactive with standards transformations, such as the Fiat-Shamir [14] or the Unruh [30] transform. This is the first non-interactive proof of parallelness.

Optimizations. For simplicity, the presentation of the proof $\mathcal{R}^*_{\mathsf{par}}$ preferred a schematic description, but it is possible to improve the protocol to make it more compact. Besides the optimizations applicable to the proof $\mathcal{P}_{\mathsf{iso}}$ described in Section 5.2, we remark that parallelness is independent of torsion images. Thus, the proofs of isogeny knowledge do not need to guarantee the correctness of torsion images to prove parallelness. However, in the OPRF context, the correctness of the torsion images revealed by the server is needed to guarantee verifiability: a malicious server might otherwise reveal incorrect torsion points to different users and use that information to match OPRF outputs to specific

interactions. Hence, the proof can be made more efficient by avoiding proving the correctness of torsion images for the commitment isogeny.

Concrete cost. The proof described in Fig. 5 adds the communication of the values w, x, y, z when $\mathsf{chall} = -1$. In that case, the prover's response requires $\log n + \log s + 4\log d + 2\lambda$ bits; when $\mathsf{chall} = 0$, the response is also larger because the points R_2, S_2 need to be communicated explicitly. The answers to the other challenge remains unchanged.
The same proof, when used for the commitment isogeny, can avoid proving correctness of the torsion images, resulting in a smaller proof. In particular, no masking values are ever revealed, and when $\mathsf{chall} = 0$ the response does not contain the points P_2, Q_2 on E_2. Setting $d \approx n \approx s \approx \sqrt[3]{n}$, we obtain that an average proof $\mathcal{P}_{\mathsf{par}}$ requires $\approx 1.71\lambda(49/9 \log p + 14/3\lambda)$ bits, while a worst-case proof would require $\approx 1.71\lambda(9 \log p + 6\lambda)$ bits.

7 A new OPRF protocol

In this section, we combine the countermeasures presented in Section 4, the SIDH countermeasures and the novel proof of isogeny knowledge discussed in Section 5, and the non-interactive proof of parallel isogeny introduced in Section 6 to obtain a verifiable OPRF protocol that is post-quantum secure, round-optimal, and moderately compact.
The OPRF protocol is a two-party protocol between a user U and a server S. Let N_{M}, N_{B}, N_{K} be coprime numbers representing the degrees of the message isogeny, the blinding isogeny, and the server's isogeny, respectively. Let p be a prime of the form $p = N_{\mathsf{M}} N_{\mathsf{B}} N_{\mathsf{K}} f - 1$, for some cofactor f, and let $E_0, \tilde{E}$ be two supersingular elliptic curves defined over $\mathbb{F}_{p^2}$. Moreover, let P, Q be a fixed basis of $E_0[N_{\mathsf{M}}]$ and let $\tilde{P}, \tilde{Q}$ be a fixed basis of $\tilde{E}[N_{\mathsf{K}}]$. The first curve is used to compute the PRF, while the second is used within the server's commitment. At a high-level, to evaluate the OPRF on an input x, the user maps the input to a curve E_m according to Algorithm 1 and computes a blinding isogeny $\phi_b : E_m \to E_{mb}$. The user then sends the codomain curve, together with torsion images and a proof of their correctness, to the server, which computes a second isogeny $\phi_k : E_{mb} \to E_{mbk}$. The torsion information is appropriately masked to avoid the SIDH attacks. The server then responds with the curve E_{mbk}, some torsion information, a proof of their correctness, and a proof that it has used the previously-committed secret key. The user then concludes by using the torsion information provided by the server to undo the blinding isogeny and compute the curve E_{mk}. Its j-invariant is then hashed together with the input and the server's public key to form the PRF output. The protocol is described in Fig. 7, and it realizes the OPRF ideal functionality of Fig. 2, which allows us to state the following theorem.

Theorem 1. *The protocol described in Fig. 7 realizes the ideal functionality $\mathcal{F}_{\mathsf{vOPRF}}$ of Fig. 2 in the random oracle model.*

The proof follows the same line as the security proof of the OPRF protocol by Boneh, Kogan, and Woo [5, Theorem 20], since the hardness assumption of Problem 4 and the proof $\mathcal{P}_{\mathsf{iso}}$ are a drop-in replacement for the Auxiliary One-More SIDH assumption and the NIZKPK proof used in [5], respectively. At a high level, the case of an honest user and malicious server in the proof is simple because the server only interacts with the user through their first query, and in that case the user's security corresponds to the input hiding property, guaranteed by the hardness of Problem 1. The case of a malicious user is more complicated, because the user has output. The server can be simulated as a honest server, but to ensure that the malicious user output is indistinguishable from the ideal-world, the random oracle $\bar{H}$ can be programmed to output the ideal-world output. This would create a problem with the ticketing system of the ideal functionality if the adversary could produce more OPRF outputs than the number of interactions, but the one-more unpredictability property prevents that. The main difference between this proof and that of [5] is the use of a non-interactive proof of parallel isogeny, which results in a simpler proof since the proof of knowledge can be simulated. Note that the proof in [5] is written in terms of the augmentable commitment abstraction, which we preferred avoiding; since the same security properties can be directly expressed in terms of the OPRF protocol, as shown in Section 3, the difference is purely syntactical.

Parameters. A prime p of the form $p = N_{\mathsf{M}} N_{\mathsf{B}} N_{\mathsf{K}} f - 1$, where $N_{\mathsf{M}}, N_{\mathsf{B}}, N_{\mathsf{K}}$ are smooth coprime integers and f a smooth cofactor. E_0 and $\tilde{E}$ are supersingular elliptic curves defined over $\mathbb{F}_{p^2}$, where $\mathrm{End}\,\tilde{E}$ is unknown, and $P, Q \in E_0[N_{\mathsf{M}}]$ and $\tilde{P}, \tilde{Q} \in E[N_{\mathsf{K}}]$ are fixed bases. The protocol also relies on the following functions:

- $H_i : \{0,1\}^* \to \mathbb{Z}_M$ for $i \in \{1, \ldots, I\}$, where I is such that $N_{\mathsf{M}}^I > 2^{4\lambda}$
- $\bar{H} : \{0,1\}^* \to \{0,1\}^\lambda$, to hash the final PRF output,

and two non-interactive proofs of knowledge: $\mathcal{P}_{\mathsf{iso}}$, for the user to prove correctness of torsion images, and $\mathcal{P}_{\mathsf{par}}$, for the server to prove it computed honestly with the committed key.

Initialization. On input INIT from the environment, the server S:

- sample $k \leftarrow \mathbb{Z}_K$ and stores it,
- computes the curve $\tilde{E}_C = \tilde{E}/\langle \tilde{P} + [k]\tilde{Q} \rangle$,
- stores $\mathsf{pk} = (j(E_C))$ and outputs (INIT, pk).

Evaluation. On input INIT from the environment, the server S:

- On input (EVAL, S, x), the user U proceeds as follows:
  1. Sample $\alpha \leftarrow \mathbb{Z}_N^*$ and $b \leftarrow \mathbb{Z}_B$,
  2. Compute $(\phi_m, E_m) = \mathcal{H}_I(x)$;
  3. Compute $\phi_b : E_m \to E_{mb} := E_m/\langle P_m + [b]Q_m \rangle$, where $P_m, Q_m = \mathcal{B}_B(E_m)$,
  4. Set $\phi_{mb} = \phi_b \circ \phi_1 \circ \phi_0$, $R = [\alpha]\phi_{mb}(P)$, $S = [\alpha]\phi_{mb}(Q)$,
  5. Compute $\pi_c \leftarrow \mathcal{P}_{\mathsf{iso}}(E_0, P, Q, E_{mb}, R, S, \phi_{mb}, \alpha)$,

6. Send message (E_{mb}, R, S, π_c) to the server and store ϕ_b.

– On input SERVERCOMPLETE from the environment and message (E_{mb}, R, S, π_c) from the user U, the server S proceeds as follows:
1. Verify the proof π_c and sample $\alpha_k \leftarrow \mathbb{Z}_n^*$,
2. Compute $\phi_k : E_{mb} \to E_{mbk} \coloneqq E_{mb}/\langle R + [k]S \rangle$,
3. Compute $R_k = [\alpha_k]\phi_k(P_b)$, $S_k = [\alpha_k]\phi_k(Q_b)$, where $P_b, Q_b = \mathcal{B}_B(E_{mb})$,
4. Compute $\pi_k \leftarrow \mathcal{P}_{\mathsf{par}}((E_{mb}, P_b, Q_b, E_{mbk}, R_k, S_k), (\tilde{E}, \tilde{P}, \tilde{Q}, \tilde{E}_C), k, \alpha_k)$[2],
5. Send $(\mathsf{pk}, E_{mbk}, R_k, S_k, \pi_k)$ to the user U.

– On input $(\mathsf{pk} = j(E_c), E_{mbk}, R_k, S_k, \pi_k)$ from the server S, the user U proceeds as follows:
1. Verify the proof π_k,
2. Compute b_0, b_1 such that $\langle [b_0]P_b + [b_1]Q_b \rangle = \ker \hat{\phi}_b$, where $P_b, Q_b = \mathcal{B}_d(E_{mb})$,
3. Compute $\phi_u : E_{mbk} \to E_{mk} \coloneqq E_{mbk}/\langle [b_0]R_k + [b_1]S_k \rangle$,
4. Compute $y = \bar{H}(x, \mathsf{pk}, j(E_m k))$ and output $(\text{EVAL}, \mathsf{pk}, y)$.

Fig. 7: The verifiable OPRF protocol.

Parameter selection. Firstly, we discuss how to select the starting curves E_0 and $\tilde{E}$. As mentioned in Section 5, the cryptanalysis on masked-torsion SIDH with a starting curve with small endomorphism [15, Section 4.2] does not apply here, since the message isogeny removes this property from the starting curve of the blinding isogeny. Hence, the curve E_0 does not need to have unknown endomorphism ring. However, the situation is different for $\tilde{E}$: as observed in [4], knowledge of $\mathrm{End}\,\tilde{E}$ allows to find collisions in the server's commitment. Thus, knowing $\mathrm{End}\,\tilde{E}$ would allow the server to break verifiability, since it could prove parallelness to two distinct isogenies. It is thus necessary that the curve $\tilde{E}$ is generated by a trusted party or through a multiparty trusted setup ceremony, such as the one presented in [3].
The main parameter of the OPRF protocol is the prime p. Firstly, if the message isogeny is the composition of many isogenies whose kernel is defined over $\mathbb{F}_{p^4}$, the value $p+1$ does not need have a dedicated factor. Then, for the main exchange, i.e. the blinding, server's isogeny, unblinding part, we need to smooth coprime integers N_B and N_K that are highly composite to prevent the SIDH attacks. Following the analysis of Section 5, we have $N_\mathsf{B} \approx N_\mathsf{K} \approx 2^{2956}$. Lastly, the proofs of knowledge $\mathcal{P}_{\mathsf{iso}}$ and $\mathcal{P}_{\mathsf{par}}$ require a third cofactor N_S that is coprime with both N_B and N_K. To guarantee the hardness of Problems 6 and 7, the integer N_S needs to be of the same length as N_B and N_K. However, since torsion points of order N_S do not need to be masked, the value N_S can be a prime power. Putting this

[2] The proof algorithm does not receive torsion points because, as discussed in Section 6, they are not necessary to prove parallelness.

together, we obtain that the prime p needs to be of the form $p = N_B N_S N_K f - 1$ and be at least 8868-bit long to guarantee $\lambda = 128$ bits of security. Note that the new computation of the message isogeny and the new proofs of knowledge has significantly reduced the size of the prime; compared to the OPRF protocol by Boneh, Kogan, and Woo, we use a prime that is 5.8× larger, while relying on an SIDH protocol with isogenies that are 9.2× longer.

Efficiency. We now estimate the communication cost of the OPRF protocol. The largest components are the non-interactive proofs of knowledge: given the analysis of the previous sections, they are less than $1.7\lambda(35 \log p + 51\lambda)$-bit long. Since $\log p \approx 10\lambda \log \lambda$, we obtain that one OPRF execution requires $1.7\lambda^2(350 \log \lambda + 51)$ bits of communication. For $\lambda = 128$, this corresponds to a transcript of 8.7 MB. We remark that the size of the proofs is particularly large due to the Unruh transform needed to prove security in the UC framework. If the proofs were made non-interactive via the Fiat-Shamir transform, a single execution of the verifiable OPRF with $\lambda = 128$ would require 1.9 MB of communication on average and 3.8 MB in the worst case. Such an OPRF may be used in instances where security in the UC framework is not necessary.
A direct comparison with the protocol by Boneh, Kogan, and Woo [5] is not simple since their bandwidth estimate does not appear to include the Unruh transform overhead. We estimate that one execution of the OPRF from [5] requires at least 10.9 MB[3]. Our protocol is thus more compact than that in [5], despite being round-optimal and secure against both the one-more unpredictability attack and the SIDH attacks. This is made possible by the fact that the sigma protocols are highly optimized and have ternary challenges, which significantly reduces the overhead introduced in the Unruh transform. Indeed, if we compare a version of the two protocols with the Fiat-Shamir transform, our OPRF uses 31% more bandwidth than the one in [5]. We summarize the state of post-quantum OPRF protocols in Table 1.

8 Conclusion

In this work, we presented a post-quantum verifiable oblivious PRF protocol that is moderately compact and round-optimal. The protocol is the first round-

[3] In [5, Section 5], the authors estimate that the largest response in the sigma protocol R_{com} requires $6 \log p + 5\lambda$ bits. The protocol has a challenge space of size 6, and it needs to be repeated 3.8λ times to obtain a negligible soundness error. Without considering the size of the committments, the Unruh-based NIZKP contains 6 hashed values that are as long as the largest response, per each iteration. The transcript of an R_{com} proof thus requires at least $3.8\lambda \times 6(6 \log p + 5\lambda)$ bits. The entire OPRF hence requires three times as much (three such proofs are used), plus $5\lambda \log p$ bits for the proof of parallel isogenies.

[4] The number of communication rounds depends on the underlying OT construction. An update to [24] suggests four rounds may be necessary.

[5] Recent work by Heimberger, Meisingseth, and Rechberger [18] suggests such construction may be insecure.

Table 1: Post-quantum OPRF protocols secure against malicious clients.

Protocol	Rounds	Bandwidth (avg.)	Verifiable	Secure
[1] (LWE)	2	>128 GB	✓	✓
[5] (CSIDH)	3[4]	424 kB	✗	✓[5]
[5] $(\text{SIDH})^{\text{FO}}$	6	1.4 MB	✓	✗
[5] $(\text{SIDH})^{\text{Unruh}}$	6	>10.9 MB	✓	✗
[This work]$^{\text{FO}}$	2	1.9 MB	✓	✓
[This work]$^{\text{Unruh}}$	2	8.7 MB	✓	✓

optimal OPRF based on isogenies, and it is several orders of magnitude more compact than the existing round-optimal protocol. To obtain this protocol, we started from an insecure protocol by Boneh, Kogan, and Woo, and we proposed an efficient countermeasure against the one-more unpredictability attack, integrated the existing SIDH countermeasures, developed a new zero-knowledge proof of isogeny that works with the SIDH countermeasures, and proposed a novel non-interactive proof of parallel isogeny that reduced the number of rounds to two.

The protocol is an important stepping stone towards fully practical post-quantum OPRFs, but its performance is hindered by the inefficiency of the SIDH countermeasures. In future work, we aim at developing more efficient solutions: a moderate reduction in the degree of the isogenies would significantly improve the efficiency of the protocol. It is also interesting to improve the proof of parallel isogeny by avoiding validating the commitment isogeny at every interaction.

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